

Thermal activation of quasiparticles and thermodynamics of fractional quantum Hall liquids

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Quasielectrons and quasiholes in the fractional quantum Hall liquids obey fractional (including nontrivial mutual) exclusion statistics. Their statistics matrix can be determined from several possible state-counting schemes, involving different assumptions on statistical correlations. Thermal activation of quasiparticle pairs and thermodynamic properties of the fractional quantum Hall liquids near fillings $1/m$ (m odd) at low temperature are studied in the approximation of generalized ideal gas. The existence of hierarchical states in the fractional quantum Hall effect is shown to be a manifestation of the exclusionary nature of the relevant quasiparticles. For magnetic properties, a paramagnetism-diamagnetism transition appears to be possible at finite temperature. [S0163-1829(98)06616-8]

I. INTRODUCTION

It is well known that quantum statistics of a particle (or elementary excitation) plays a fundamental role in determining statistical or thermodynamic properties of a quantum many-body system. Bose-Einstein and Fermi-Dirac statistics are two well-established ones, which are central to many familiar or novel phenomena involving many particles. For example, superfluidity or superconductivity is essentially due to Bose-Einstein condensation; and the stability of macroscopic matter is known to depend crucially on the Fermi-Dirac statistics of electrons. Since the early days of quantum mechanics, an outstanding problem has been to search for a *generalization* of or even an *interpolation* between these two statistics. Mathematically, of course, there exist many possibilities. But as physicists we are interested in what are *physically relevant*, in the sense that the new statistics must be realized in physical systems existing in nature (or at least, in models that describe some interesting aspects of real physics).

In the last two decades or so, the interest in this search has become stronger and stronger in the study of lower-dimensional condensed-matter systems. For quite a while, it has been recognized that situations that interpolate between bosons and fermions may appear in one-^{1,2} and two-dimensional³ many-body systems, though there has been no discussion of relevant statistical distributions until recently.

By now there have been (at least) two distinct ways to define fractional statistics: (1) by examining the change in phase of a multiparticle wave function due to the exchange of two identical particles, (2) by counting the number of independent multiparticle quantum states to formulate a gen-

eralization of Pauli exclusion principle. For the usual quantum statistics, i.e., for bosons and fermions, the above two methods are equivalent to each other, in spite of their conceptual difference. For fractional statistics, however, they are generally inequivalent and we distinguish between the two definitions by calling them “exchange statistics” and “*exclusion statistics*,” respectively, and the corresponding particles “anyons” and “*exclusons*.” Fractional exchange statistics or anyons³ have been first explored in the study of quasiparticles in two- (space) dimensional systems, such as fractional quantum Hall (FQH) liquids and anyon superconductivity.

Recently Haldane⁴ has formulated, by counting many-body states, a generalized Pauli exclusion principle in arbitrary spatial dimensions. Based on this idea, Wu⁵ has defined a generalized ideal gas for particles obeying such fractional (including mutual) exclusion statistics, and has formulated its quantum statistical mechanics and thermodynamics. These new definitions are not merely mathematical construction; they have been shown to be realized as the exotic statistics obeyed by elementary excitations in certain one-, two-, or higher-dimensional strongly correlated systems.^{4–11} In this paper, we discuss an important case: quasiparticles in the fractional quantum Hall (FQH) effect.

By now it is well known that the ground state of the two-dimensional electron gas in a strong perpendicular magnetic field with electron filling factor $\nu = 1/m$ (m odd integer) is an incompressible quantum liquid,¹² and that it has two species of quasiparticle excitations, both of which are fractionally charged¹² ($e_{\pm}^* = \pm 1/m$) anyons.^{13,14} However, the anyon approach is not very suitable for calculating low-temperature thermodynamic properties of the FQH liquids, since thermal activation of quasiparticle pairs is directly gov-

erned by the counting law for many-body states rather than the law for exchange phases. This is where the concept of exclusion statistics comes into play. The FQH quasiparticles are known to be strongly correlated. The key issue here is to clarify how strong correlation between quasiparticles manifests itself in state counting. Not only is fractional exclusion of single-particle states for identical quasiparticles possible, but also mutual exclusion of states between a quasihole and a quasielectron, which cannot be dealt with in the anyon approach.

The many-body state counting for FQH quasiparticles is a subtle problem, whose study started with Haldane's paper in 1983.¹⁵ Now we still do not have a full answer, except for the low-lying excited states of the FQH liquids, which, however, should be enough to account for low-temperature thermodynamic properties. There are several possible assignments for the statistics matrix of FQH quasiparticles, involving different assumptions concerning the nature of the quasiparticles and their statistical correlations. (See below, Sec. III, for details.) Fortunately, these assignments of a statistics matrix can be tested by numerical simulation on small systems. In addition, one would also like to put these assignments to experimental tests, which requires comparing theoretical predictions with experimental data. As a first step towards this, we have calculated thermodynamic properties of the FQH liquids at low temperatures, based on the dilute gas approximation for thermally activated quasiparticles: When the quasiparticles are dilute, we may ignore Coulomb interactions between them, and apply the statistical thermodynamic formalism for a generalized ideal gas given in Ref. 5, which incorporates mutual statistics between different species of quasiparticles. It is hoped that this approximation could be improved in the future by including the effects of Coulomb interactions between quasiparticles. Our thermodynamic calculation is done with three different assignments of statistics matrix, with the hope that one day the experimental measurements of thermodynamic properties of FQH liquids might distinguish between them, providing information about the statistical correlations between FQH quasiparticles.

This paper is organized as follows. We first review in Sec. II the state-counting definitions of exclusion statistics, including mutual statistics between nonidentical (quasi)particles. Then in Sec. III it is shown that quasielectrons and quasiholes in the FQH liquids with fillings $1/m$ (m odd) obey such fractional exclusion statistics, and the determination of their statistics matrix from various physical arguments or working hypotheses is reviewed, with several somewhat different outcomes. Using the statistical distribution⁵ for a generalized ideal gas, we study in Secs. IV and V, with analytic and numerical methods, thermal activation of quasiparticle pairs near filling $1/m$ (m odd). Then we show in Sec. VI that the occurrence of new (hierarchical) incompressible states, corresponding to divergent pressure, at appropriate fillings at $T=0$ is a manifestation of fractional exclusion statistics, while at finite temperature the pressure of the system can never become divergent. In Sec. VII we compute low-temperature thermodynamic properties of FQH liquids, including magnetic properties. The final section VIII is devoted to conclusions and discussions.

II. EXCLUSION STATISTICS

The definition of fractional exclusion statistics is directly based on state counting, a basic concept in quantum statistical mechanics. It is well known that bosons and fermions have different counting for many-body states, or different statistical weight W : The number of quantum states of N identical particles occupying a group of G states, for bosons or fermions, respectively, is given by

$$W_b = \frac{(G+N-1)!}{N!(G-1)!}, \quad \text{or} \quad W_f = \frac{G!}{N!(G-N)!}. \quad (2.1)$$

A simple generalization and interpolation is

$$W = \frac{[G + (N-1)(1-\alpha)]!}{N![G - \alpha N - (1-\alpha)]!}, \quad (2.2)$$

with $\alpha=0$ corresponding to bosons and $\alpha=1$ fermions. The physical meaning of this equation is the following: By assumption, the statistical weight remains to be a *single combinatoric number*, so one can count the states by thinking of the particles *effectively as bosons*, with the effective number of available single-particle states being *linearly dependent on the particle number*:

$$G_{\text{eff}}^{(b)} = G - \alpha(N-1). \quad (2.3)$$

Obviously, for genuine bosons, $G_{\text{eff}}^{(b)}$ is independent of the particle number. In all other cases, $G_{\text{eff}}^{(b)}$ is linearly dependent on the particle number. This is the defining feature of the fractional exclusion statistics. The statistics parameter α tells us, on the average, how many single-particle states that a particle can exclude others to occupy. For $\alpha \neq 1$, this generalizes the Pauli exclusion principle for one species.

It is easy to generalize this state counting to more than one species:

$$W = \prod_i \frac{[G_i + N_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})]!}{(N_i)! [G_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})]!}. \quad (2.4)$$

Here G_i is the number of states when the system consists of only a single particle. By definition, the diagonal α_{ii} is the "self-exclusion" statistics of species i , while the non-diagonal α_{ij} (for $i \neq j$) is the mutual-exclusion statistics. Note that α_{ij} , which Haldane⁴ called *statistical interactions*, may be *asymmetric* in i and j . The interpretation is similar to the one species case. The number of available single-particle states for species i , in the presence of other particles, is again linearly dependent on particle numbers of all species:

$$G_{\text{eff},i}^{(b)} = G_i - \sum_j \alpha_{ij}(N_j - \delta_{ji}). \quad (2.5)$$

We note that as a generalized Pauli exclusion principle, Eq. (2.2) or (2.4) implies strong correlations between the particles, and does not give rise to the same state counting as the old generalization suggested in Ref. 16, in which particles independently fill a fixed number of single-particle states with the constraint that at most n particles are allowed in one and the same state.

Some remarks on exclusion statistics are in order:

(1) This definition of exclusion statistics is *independent of spatial dimensionality* of the system, in contrast to the exchange statistics of anyons, which has a connection to the braid group, making sense only in two spatial dimensions.¹⁷

(2) In contrast to anyons, there is *no periodicity* in exclusion statistics parameter α , so it makes sense to consider the cases with $\alpha > 1$ or even $\alpha > 2$.

(3) The state-counting definition of exclusion statistics naturally allows *mutual statistics* from the beginning, implying that exclusion may occur between states of different species, a completely new situation we have not been faced before in statistical mechanics.

III. EXCLUSION STATISTICS FOR FQH QUASIPARTICLES

There are two kinds of quasiparticles in the Laughlin $1/m$ liquid: quasiholes labeled by $-$ and quasiparticles labeled by $+$. In this paper we treat them as two distinct species and demonstrate that their many-body states obey the counting law given by Eq. (2.5), with an appropriate 2×2 statistics matrix α_{ij} with $i, j = +, -$. The statistics matrix depends on the nature of FQH quasiparticles and their correlations. Several scenarios are possible in this regard. In this section, we are going to discuss four possible scenarios for FQH quasiparticles that have appeared in the literature: (1) anyons in the lowest Landau level, (2) bosonic vortices, (3) composite fermions, and (4) correlated vortices or composite fermions. They differ in the assumption of whether certain correlations, such as hard-core constraints, exist between the quasiparticles or not, leading to subtle difference in statistics matrix. It is remarkable that the statistics matrix can be subject to a numerical test for small systems on a sphere. We are not going to talk about the details, but will briefly summarize the status of such numerical tests and quote relevant references when appropriate.

A. Anyons in the lowest Landau level

In determining exclusion statistics of FQH quasiparticles, let us first try to explore the fact that they are fractionally charged anyons. (Though later we will see that the picture of noninteracting anyons is not very suitable for calculating thermal activation of quasiparticle pairs.)

Good trial electron wave functions for states with quasiparticles in the $1/m$ FQH liquid were first proposed by Laughlin.¹² In these wave functions the coordinates of the quasiparticles appear as parameters (or collective coordinates). If one moves very slowly the coordinates of one quasiparticle, say a quasihole, around a closed loop in the FQH liquid, the electron wave function acquires a Berry phase, which can be interpreted as the phase due to the motion of the quasiparticle traveling along the loop. As shown in Ref. 14, the Berry phase is always proportional to the number of electrons enclosed in the loop. If the loop encloses none of other quasiparticles, the Berry phase is the same as that for a charge in a magnetic field, with electrons acting as quantized sources of “flux.” Thus, a quasiparticle sees the electrons just like an electron sees the external magnetic field. When the loop encloses another quasiparticle, say a quasihole, the change in the Berry phase is due to a deficit in the number of enclosed electrons caused by the enclosed quasihole, and it is

attributed to the exchange phase of the two quasiholes, showing that they are anyons with fractional exchange statistics.¹⁴ Combining the two results, one is led to a simple picture that the FQH quasiparticles are anyons in the lowest Landau level of a fictitious magnetic field, whose strength is determined by the density of electrons. Indeed, the wave function for the FQH quasiparticles suggested by Halperin¹⁸ are such as if the quasiparticles are in the lowest Landau level.

Now let us count the states of N noninteracting anyons (of one species) in the lowest Landau level. Though not for all levels and all states, a number of exact solutions for anyons in a magnetic field have been known.¹⁹ Among them, fortunately, are the complete set of solutions for all anyons in the lowest Landau level (if the number of anyons is less than the Landau degeneracy). The total (ground state) energy turns out to be the sum of the cyclotron energy of individual particles, independent of the exchange statistics of anyons:

$$E = N\varepsilon_c/2. \quad (3.1)$$

To count the states, we consider anyons in a circular disk with a fixed size. In the symmetric gauge, besides the usual Gaussian factor, the many-anyon wave function is known to be of the form (with z_i the complex coordinates of electrons)

$$\Psi = \prod_{i < j} (z_i - z_j)^{\theta/\pi} \cdot \Phi(z_1, \dots, z_N), \quad (3.2)$$

with anyon statistics $0 \leq \theta < 2\pi$. But now in the lowest Landau level, the function Φ is a symmetric polynomial of (z_1, \dots, z_N) . The state counting can be easily done by looking at the symmetric polynomial Φ (counting as bosons). However, the fixed-size condition requires a fixed highest angular momentum, or a fixed highest power of a single variable z_i in the wave function Ψ . On the other hand, the Jastrow-type prefactor $\prod_{i < j} (z_i - z_j)^{\theta/\pi}$, implying nonvanishing relative angular momenta between anyons, takes away some powers of z_i and reduces the degree of the polynomial Φ . Alternatively, an anyon can see the statistical flux of other anyons, which in the present case is opposite to the external magnetic flux. Therefore, in the boson counting, *with size and external flux fixed*, the effective Landau degeneracy is determined by the external magnetic flux N_ϕ less the anyon statistical flux $(\theta/\pi)(N-1)$.²⁰

$$G_{\text{eff}}^{(b)} = \frac{N_\phi}{m} - \frac{\theta}{\pi}(N-1). \quad (3.3)$$

Hence Eq. (2.2) applies, with the single anyon Landau degeneracy $G = N_\phi/m$, and the exclusion statistics for anyons in the lowest Landau level can be read off from Eq. (2.3):

$$\alpha = \theta/\pi. \quad (3.4)$$

In Ref. 14, the exchange statistics has been shown to be $\theta_- = \pi/m$ for quasiholes, and $\theta_+ = -\pi/m$ for quasielectrons. Thus, Eq. (3.4) leads to the following diagonal exclusion statistics for quasiparticles:

$$\alpha_{--} = \frac{1}{m}, \quad \alpha_{++} = \begin{cases} -1/m & (\text{soft core}), \\ 2 - 1/m & (\text{hard core}). \end{cases} \quad (3.5)$$

For the case of the quasielectron, we note that there are two possibilities: Since conceptually exchange statistics is an angular parameter, defined only up to a period of 2π , $\theta = -\pi/m$ is equivalent to $\theta = (2 - 1/m)\pi$. However, exclusion statistics is always unambiguously defined and nonperiodic at all; i.e., the exclusion effects with $\alpha = -1/m$ and with $\alpha = 2 - 1/m$ are very different. So when one wants to apply Eq. (3.4), one has to choose between the two possible values of θ . One may notice that the wave function (3.2) with $\theta = -\pi/m$ is singular at $z_i = z_j$. Based on the braid group, one may argue that the many-anyon wave function (2.1) should vanish as two anyons approach each other, thus preferring $\theta/\pi = 2 - 1/m$ over $-1/m$ in the Jastrow-type prefactor and therefore $\alpha = 2 - 1/m$ over $-1/m$ for the exclusion statistics for quasielectrons. Obviously, the former value of α leads to stronger exclusion between quasielectrons, as if they have “hard core.”

Whether the quasielectrons really satisfy the “hard-core constraint” can be tested by numerical experiments. Such numerical experiments have been done by three groups (Refs. 21–22) for electrons with Coulomb interactions on a sphere in the field of a monopole at its center. Their results unambiguously support the exclusion statistics (3.5) with the “hard-core” value for quasielectrons.

B. Bosonic vortex scheme

The above scenario for FQH quasiparticles as anyons in the lowest Landau level has the disadvantage that it tells us nothing about the mutual (or nondiagonal) statistics between quasihole and quasielectron, which is important for studying thermal activation of quasiparticle pairs. So we need other, more direct ways to count states for quasiparticles in the Laughlin $1/m$ liquid.

A fundamental relation that will play a key role in state counting is the “total-flux” constraint

$$N_\phi \equiv eBV/hc = mN_e + N_- - N_+, \quad (3.6)$$

between the electron numbers N_e and quasiparticle numbers N_- and N_+ . The basic idea behind this relation is the following observation:¹² To generate a quasiparticle in the incompressible Laughlin liquid, one may pierce the droplet by an infinitely thin solenoid and slowly turn on magnetic flux inside it. When the flux reaches a flux quantum, a quasiparticle will be formed around the solenoid; whether it is a quasihole or a quasielectron depends on the direction of the solenoid flux (parallel or antiparallel to the external magnetic field).

Further state counting relies on the assumptions on statistical correlations of quasiparticles. There are different counting schemes based on the bosonic vortex picture, the composite fermion picture, and variation of both. Let us consider them in turn.

The bosonic vortex scheme is based on the picture that the FQHE quasiparticles are vortexlike excitations in the incompressible planar quantum liquid, and for a fixed number of excitations we count their states as if they are bosons. Assuming only the minimal (quantized) circulation, there are two possible orientations for vortex circulation on the plane, corresponding to the quasihole and the quasielectron, respec-

tively. So what is essential to this counting scheme is to determine the number of available states for each species of vortices.

This can be inferred from an observation by Haldane and Wu²³ that for vortices in a planar quantum liquid, their core X and Y coordinates do not commute with each other, as if they were the guiding-center coordinates for a charged particle in a magnetic field, with fluid particles (i.e., electrons in the present case) as sources of quantized flux. Thus, the number of available states for vortexlike excitations is essentially determined by the “Landau degeneracy” of this fictitious magnetic field, or the number of electrons:

$$G_{\text{eff},-}^{(b)} = N_e, \quad G_{\text{eff},+}^{(b)} = N_e. \quad (3.7)$$

To derive the exclusion statistics for FQH quasiparticles, one needs to fix the external flux N_ϕ . So let us express $G_{\text{eff},\pm}$ in terms of N_ϕ by eliminating N_e from these equations with the help of the constraint (3.6). Then we obtain

$$G_{\text{eff},-}^{(b)} = \frac{1}{m}N_\phi - \frac{1}{m}N_- + \frac{1}{m}N_+, \quad (3.8)$$

$$G_{\text{eff},+}^{(b)} = \frac{1}{m}N_\phi - \frac{1}{m}N_- - \left(-\frac{1}{m}\right)N_+.$$

The first term on the right side gives the single-quasiparticle degeneracy in terms of the external flux:

$$G_+ = G_- = (1/m)N_\phi, \quad (3.9)$$

so the proportionality constant $1/m$ is identified as the fractional charge (absolute value) of the quasiparticles. And the coefficients of N_\pm give the statistics matrix:

$$\begin{aligned} \alpha_{++} &= -1/m, & \alpha_{+-} &= 1/m, \\ \alpha_{-+} &= -1/m, & \alpha_{--} &= 1/m. \end{aligned} \quad (3.10)$$

This result was first derived by Haldane.⁴

Comparing with Eq. (3.5), we note that $\alpha_{++} = -1/m$ here corresponds to soft-core quasielectrons. For hard-core quasielectrons, Eq. (3.7) should be replaced by

$$G_{\text{eff},-}^{(b)} = N_e, \quad G_{\text{eff},+}^{(b)} = N_e - 2(N_+ - 1), \quad (3.11)$$

with the second term in $G_{\text{eff},+}^{(b)}$ representing the exclusion effects due to the *hard core* of quasielectrons. In the presence of an external magnetic field, the two orientations of vortex circulation are not equivalent, so there is an asymmetry between quasiholes and quasielectrons. Physically, the hard-core nature of quasielectrons may be due to electron number surplus in the core of quasielectrons. This leads to²⁴

$$G_{\text{eff},-}^{(b)} = \frac{1}{m}N_\phi - \frac{1}{m}N_- + \frac{1}{m}N_+, \quad (3.12)$$

$$G_{\text{eff},+}^{(b)} = \frac{1}{m}N_\phi - \frac{1}{m}N_- - \left(2 - \frac{1}{m}\right)N_+,$$

resulting in a statistics matrix somewhat different from Eq. (3.10):

$$\begin{aligned}\alpha_{++} &= 2 - 1/m, & \alpha_{+-} &= 1/m, \\ \alpha_{-+} &= -1/m, & \alpha_{--} &= 1/m.\end{aligned}\quad (3.13)$$

We note that now the diagonal statistics α_{++} in Eq. (3.13) agree with that of hard-core quasielectrons in Eq. (3.5). However, here it has been possible to demonstrate nontrivial mutual statistics between quasihole and quasielectron. This means that with N_ϕ fixed, the presence of quasielectrons will affect the number of “available” states for quasiholes and *vice versa*.

C. Composite fermion scheme

The central idea of the composite fermion approach is that the FQH state of electrons in a physical magnetic field can be explained as the IQH state of composite fermions in an effective magnetic field.²⁵ Imagine an adiabatic process in which we somehow collect $2p$ (p an integer) flux quanta to each electron to form an electron-flux composite. The additional Aharonov-Bohm phase, due to the attached flux, associated with moving one composite around another is $e^{i2p\pi} = 1$. So the statistics of the composite remains the same as the electron, motivating the name composite fermion. These composites are now moving in a reduced magnetic field $B_{\text{eff}} = B - 2\pi(2p\rho)$, where ρ is the density of electrons, which is the same as the density of composite fermions. (Recall that in our convention, the unit of flux is 2π .) The filling factor for composite fermions then increases to ν_{eff} , given by $\nu_{\text{eff}}^{-1} = (B - 4\pi\rho)/2\pi\rho = \nu^{-1} - 2p$. For $\nu_{\text{eff}} = n$ (n an integer), we have

$$\nu = \frac{n}{2pn + 1}. \quad (3.14)$$

Thus, fractional Hall systems with $\nu = n/(2pn + 1)$ may be adiabatically changed into an integer Hall system with filling factor n , as was also emphasized by Greiter and Wilczek.²⁶ Note that this argument gives us more than we had hoped for. The case we wanted to understand, with $\nu^{-1} = \text{odd}$, is obtained for $n = 1$.

Let us do state counting for the state with $n = 1$, or $\nu = 1/(2p + 1)$. The Landau degeneracy for the composite fermion in the residue magnetic field is given by the effective flux

$$N_{\phi, \text{eff}} = N_\phi - 2pN_e, \quad (3.15)$$

while the number of excitations are determined by

$$N_{\phi, \text{eff}} = N_e + N_- - N_+. \quad (3.16)$$

Eliminating $N_{\phi, \text{eff}}$ from these two equations, we recover the same constraint (3.6) as before with $m = 2p + 1$.

The number of available single-particle states for unit-charged composite-fermion excitations is obviously

$$G_{\text{eff}, \mp} = N_{\phi, \text{eff}} - (N_{\mp} - 1). \quad (3.17)$$

Here we have assumed that the magnetic field is so strong that we can ignore the possibility for the quasielectron to fill Landau levels higher than the lowest available one.

To derive the true charge and statistics of the quasiparticle excitations, we need to express $N_{\phi, \text{eff}}$ in Eq. (3.17) in terms of the external N_ϕ , resulting in²⁴

$$G_{\text{eff}, -}^{(b)} = \frac{1}{m}N_\phi - \frac{1}{m}N_- - \left(1 - \frac{1}{m}\right)N_+, \quad (3.18)$$

$$G_{\text{eff}, +}^{(b)} = \frac{1}{m}N_\phi + \left(1 - \frac{1}{m}\right)N_- - \left(2 - \frac{1}{m}\right)N_+.$$

Here we have used Eqs. (3.15) and (3.16). The coefficient of N_ϕ recovers the fractional charge (absolute value) $1/m$ for the quasiparticles; see Eq. (3.9). From the coefficient of the other terms one reads off the exclusion statistics:

$$\begin{aligned}\alpha_{++} &= 2 - 1/m, & \alpha_{+-} &= -1 + 1/m, \\ \alpha_{-+} &= 1 - 1/m, & \alpha_{--} &= 1/m.\end{aligned}\quad (3.19)$$

D. Correlated vortex or projected composite fermion scheme

Comparing Eq. (3.19) with Eq. (3.13), we see that the composite fermion scheme leads to the same diagonal statistics both for quasiholes and for quasielectrons as the bosonic vortex scheme, with quasielectrons being automatically *hard core*. Thus it is not surprising that the two schemes give the same prediction about the occurrence of hierarchical states at $T = 0$, since only one species of quasiparticles exist at $T = 0$ when the filling factor deviates from the magic $1/m$, so that only diagonal statistics is relevant.

However, mutual statistics in Eqs. (3.19) and (3.13) obtained from the above two schemes are obviously different. Which is correct? Or is either one correct? To decide, one needs to study situations in which both species of quasiparticles coexist at the same time. This problem has been numerically studied in Ref. 27 (see also Ref. 29). It turns out that neither of the mutual statistics given in Eqs. (3.13) and (3.19) is correct. The correct statistics matrix, for low-lying excitations, turns out to be

$$\begin{aligned}\alpha_{++} &= 2 - 1/m, & \alpha_{+-} &= -2 + 1/m, \\ \alpha_{-+} &= 2 - 1/m, & \alpha_{--} &= 1/m.\end{aligned}\quad (3.20)$$

In the bosonic vortex scheme, this can be obtained by incorporating a certain amount of mutual exclusion (or inclusion) between vortices and antivortices in Eq. (3.11) as follows:

$$G_{\text{eff}, -}^{(h)} = N_e - 2N_+, \quad (3.21)$$

$$G_{\text{eff}, +}^{(b)} = N_e + 2N_- - 2(N_+ - 1).$$

We call this scheme the correlated vortex scheme, since in Ref. 27 it has been shown that this modification is due to the necessity of inserting some “hard-core” Jastrow factor between quasihole and quasielectron in the quasiparticle wave functions, which represents correlations of a new type between the vortex and the antivortex.

To reproduce the statistics (3.20) in the composite fermion scheme, one needs to modify Eq. (3.17) to^{28,29}

$$G_{\text{eff},-} = N_{\phi,\text{eff}} - (N_- - 1) - N_+,$$

$$G_{\text{eff},+} = N_{\phi,\text{eff}} + N_- - (N_+ - 1). \quad (3.22)$$

The mutual exclusion added in these equations between composite fermionic holes and composite fermionic electrons can be interpreted as a consequence of the *projected* composite fermion scheme: In the composite fermion transformation quasielectron states involve wave functions in the second Landau level, so it is necessary to project the wave functions down to the lowest Landau level to obtain the correct many-electron wave functions. (For details, see Ref. 30.) Indeed, the state counting resulting from the above equation has been checked^{28,29} to be indeed in agreement with the numerical data given in Ref. 30, which verifies the necessity for the projected composite fermion scheme.

We note that in either scheme, the mutual statistics between quasihole and quasiparticle are *antisymmetric* rather than symmetric.

In summary, the bosonic vortex scheme (3.11) and the unprojected composite fermion scheme (3.17) lead to different mutual statistics (3.13) and (3.19), respectively. But the correlated vortex scheme (3.21) and the projected composite fermion scheme (3.22) lead to the same statistics matrix (3.20). Numerical data favor the latter. But we feel that there is no harm in leaving these possibilities open to experimental tests. In the following, we calculate thermodynamic properties of FQH liquids with the three statistics matrices, with the hope that someday experiments might be able to distinguish between them.

IV. THERMAL ACTIVATION OF FQH QUASIPARTICLE PAIRS

It is well known that the low-temperature thermodynamics of a many-body system is determined by its low-lying excited states above the ground state. For FQH liquids at hand, our fundamental assumption is that their *low-lying* excited states are dominated by *weakly coupled* quasiparticles. Thus their low-temperature thermodynamic properties are determined by *thermal activation of FQH quasiparticle pairs*. At low temperatures, when the activated pairs are not very dense, one may ignore their interaction energies. Then the densities ρ_{\pm} of the excitations should be determined by the laws for a generalized ideal gas⁵ with two species, in which the following two conditions are satisfied: (1) The state-counting (2.4) for statistical weight W is applicable. (2) The total energy (eigenvalue) is always of the form of a simple sum, in which the i th term is linear in the particle number N_i :

$$E = \sum_i N_i \varepsilon_i, \quad (4.1)$$

with ε_i identified as the energy of a quasihole ($i = -$) or a quasielectron ($i = +$). Though this condition (4.1) is very natural for *weakly* interacting FQH quasiparticles, we note it is not compatible with noninteracting anyons, except for *only one species* of anyons all in the lowest Landau level [see Eq. (3.1)]. This problem does not exist in the theoretical framework of exclusion statistics: The condition (4.1) is compatible with free exclusions, as exemplified⁶⁻¹¹ in one-

dimensional exactly solvable many-body models such as the δ -function repulsive boson gas¹ and the Calogero-Sutherland model.² This is one of the main theoretical advantages of exclusion statistics over exchange (or anyon) statistics in dealing with statistical mechanics. (Moreover, the anyon picture cannot deal with more than one species, so it is not suitable for studying thermal activation, which involves both quasielectrons and quasiholes and is expected to be a good place to look for the effects of mutual statistics, with increasing density of activated pairs.)

With these assumptions, now we are able to derive quantum statistical mechanics of FQH quasiparticles. Consider a grand canonical ensemble at temperature T and with chemical potential μ_i for species $i = +, -$, whose partition function is given by (with k the Boltzmann constant)

$$Z = \sum_{\{N_i\}} W(\{N_i\}) \exp \left\{ \sum_{i=+,-} N_i (\mu_i - \varepsilon_i) / kT \right\}. \quad (4.2)$$

As usual, we expect that for very large N_i , the summand has a very sharp peak around the set of most-probable (or mean) particle numbers $\{\bar{N}_i\}$. Using the Stirling formula and introducing the average “occupation number per state” defined by $n_i \equiv \bar{N}_i / G_i$, from the maximum condition

$$\frac{\partial}{\partial n_i} \left[\ln W + \sum_{j=+,-} G_j n_j (\mu_j - \varepsilon_j) / kT \right] = 0, \quad (4.3)$$

one obtains the equations determining the most probable distribution of n_i :

$$\sum_{j=+,-} [\delta_{ij} w_j(T) + \alpha_{ij}] n_j(T) = 1, \quad (4.4)$$

with $w_i(T)$ being determined by the functional equations

$$w_+^{\alpha_{++}} (1 + w_+)^{1 - \alpha_{++}} \left(\frac{w_-}{1 + w_-} \right)^{\alpha_{+-}} = e^{(\varepsilon_+ - \mu_+) / kT},$$

$$w_-^{\alpha_{--}} (1 + w_-)^{1 - \alpha_{--}} \left(\frac{w_+}{1 + w_+} \right)^{\alpha_{-+}} = e^{(\varepsilon_- - \mu_-) / kT}. \quad (4.5)$$

From Eqs. (4.4) and (4.5), n_{\pm} are expressed in terms of w_{\pm} as

$$n_+(T) = \frac{\rho_+(T)}{\rho_0} = \frac{w_- + \alpha_{--} - \alpha_{+-}}{(w_+ + \alpha_{++})(w_- + \alpha_{--}) - \alpha_{+-} \alpha_{-+}}, \quad (4.6)$$

$$n_-(T) = \frac{\rho_-(T)}{\rho_0} = \frac{w_+ + \alpha_{++} - \alpha_{-+}}{(w_+ + \alpha_{++})(w_- + \alpha_{--}) - \alpha_{+-} \alpha_{-+}},$$

where $\rho_0 \equiv G_{\pm} / V$, and $\rho_+(T)$ and $\rho_-(T)$ are the density of quasielectrons and quasiholes, respectively. The ratio $R(T)$ of the numbers of quasielectrons and quasiholes is given by

$$R(T) \equiv \frac{n_+(T)}{n_-(T)} = \frac{w_- + \alpha_{--} - \alpha_{+-}}{w_+ + \alpha_{++} - \alpha_{-+}}. \quad (4.7)$$

According to charge conservation, only quasielectron-quasihole pairs are thermally activated, since they have op-

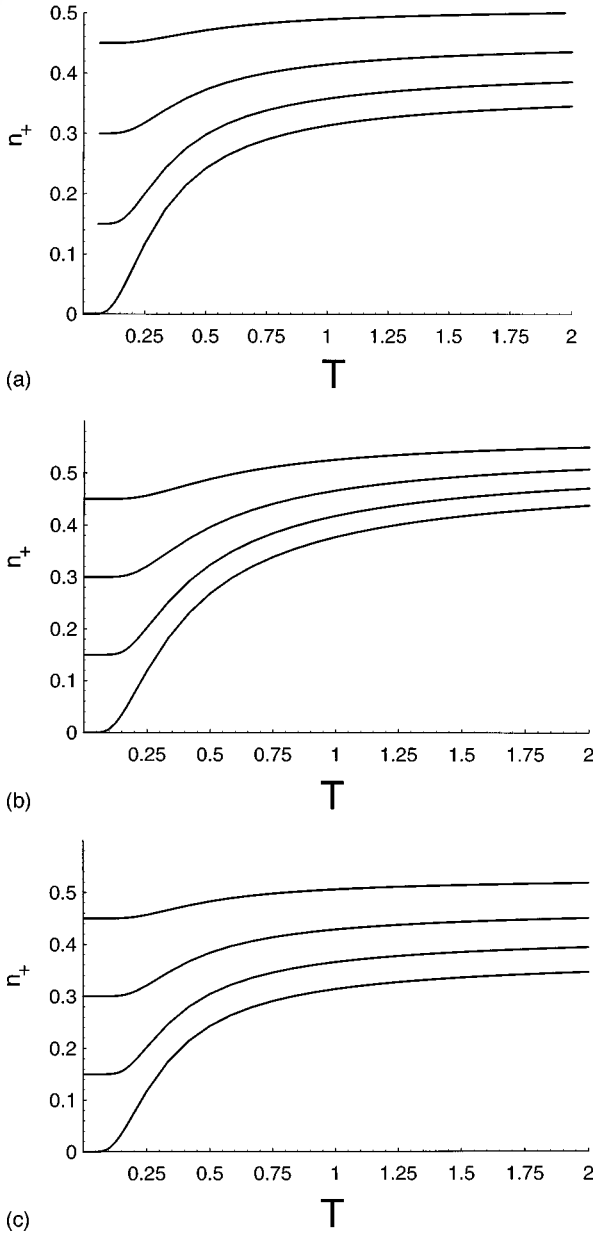


FIG. 1. The two-dimensional plot of the occupation number of the quasielectron $n_+(T, \delta)$. The filling factor is near $1/m = 1/3$. (a) For the bosonic vortex scheme, (b) for the composite fermion scheme, (c) for the correlated vortex or projected composite fermion scheme. In all the figures, $\delta = 0.15, 0.1, 0.05$, and 0 , respectively, from above.

posite charges. Thus, $N_+ - N_-$ is independent of temperature. Then the total-flux constraint (3.6) implies that

$$n_+(T) - n_-(T) = m\delta, \quad (4.8)$$

where $\delta = m(\nu - \nu_0)$, ($\nu_0 = 1/m$). Thus δ/m gives the deviation of the filling from $1/m$. Then from Eq. (4.6) one has

$$m\delta = \frac{w_- - w_+ + \alpha_{--} - \alpha_{+-} + \alpha_{-+} - \alpha_{++}}{(w_+ + \alpha_{++})(w_- + \alpha_{--}) - \alpha_{+-}\alpha_{-+}}. \quad (4.9)$$

Charge conservation also requires that the chemical potentials for the two species should satisfy $\mu_+ + \mu_- = 0$. Multi-

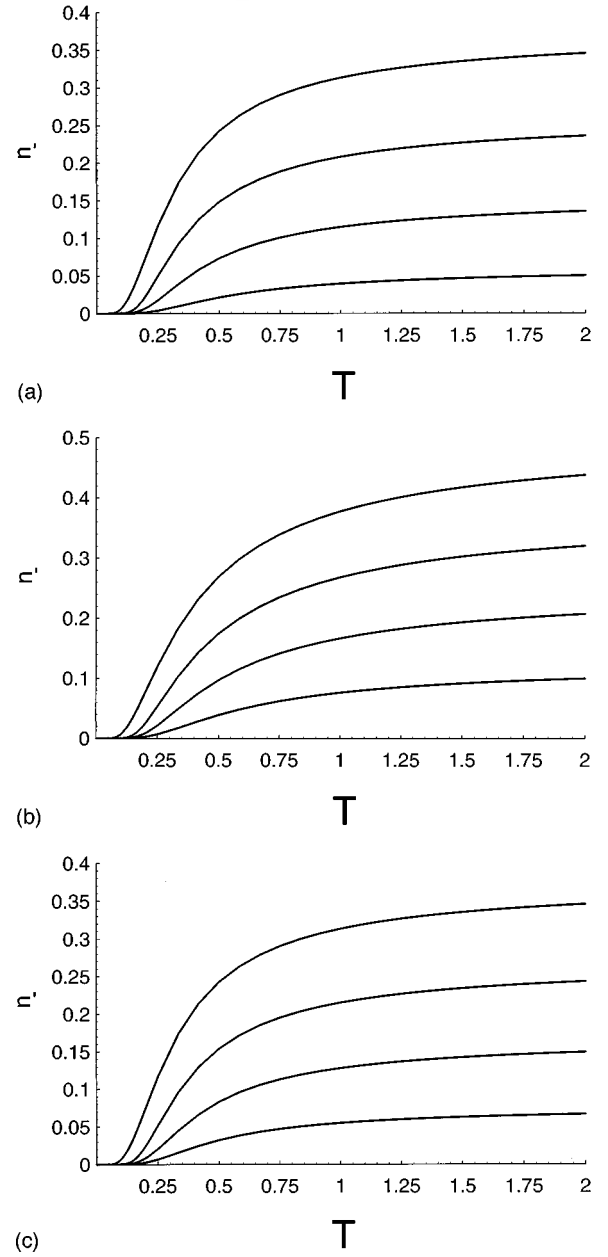


FIG. 2. The two-dimensional plot of the occupation number of the quasihole $n_-(T, \delta)$. The filling factor is near $1/m = 1/3$. (a) For the bosonic vortex scheme, (b) for the composite fermion scheme, (c) for the correlated vortex or projected composite fermion scheme. In all the figures, $\delta = 0, 0.05, 0.1$, and 0.15 , respectively, from above.

plying the two equations in Eq. (4.5), and using the above constraints, one can derive a polynomial equation

$$w_+^{\alpha_{++} + \alpha_{+-}} w_-^{\alpha_{-+} + \alpha_{--}} (1 + w_+)^{1 - \alpha_{++} - \alpha_{+-}} \times (1 + w_-)^{1 - \alpha_{-+} - \alpha_{--}} = e^{(\epsilon_+ + \epsilon_-)/kT}. \quad (4.10)$$

Once w_+ and w_- are determined from Eqs. (4.9) and (4.10), the T -dependent densities of both species $\rho_+ = n_+ \rho_0$ and $\rho_- = n_- \rho_0$ are given by Eq. (4.6). The mutual-statistics-dependent thermodynamic potential $\Omega = -kT \ln Z$ and entropy S are

$$\Omega \equiv -PV = -kT \sum_{i=+,-} G_i \ln \frac{\rho_0 + \rho_i - \sum_{j=+,-} \alpha_{ij} \rho_j}{\rho_0 - \sum_{j=+,-} \alpha_{ij} \rho_j}, \quad (4.11)$$

$$\frac{S}{k} = \sum_{i=+,-} G_i \left[n_i \frac{\varepsilon_i - \mu_i}{kT} + \ln \frac{\rho_0 + \rho_i - \sum_{j=+,-} \alpha_{ij} \rho_j}{\rho_0 - \sum_{j=+,-} \alpha_{ij} \rho_j} \right]. \quad (4.12)$$

Further the total entropy is written as $S = \sum_i N_i s_i$ with

$$\begin{aligned} \frac{s_i}{k} = & \left[1 + \frac{\rho_0}{\rho_i} - \sum_j \alpha_{ij} \frac{\rho_j}{\rho_i} \right] \ln \left[1 + \sum_j (\delta_{ij} - \alpha_{ij}) \frac{\rho_j}{\rho_0} \right] - \ln \frac{\rho_i}{\rho_0} \\ & - \left(\frac{\rho_0}{\rho_i} - \sum_j \alpha_{ij} \frac{\rho_j}{\rho_i} \right) \ln \left(1 - \sum_j \alpha_{ij} \frac{\rho_j}{\rho_0} \right). \end{aligned} \quad (4.13)$$

Other thermodynamic functions, such as specific heat and magnetization per unit area, follow straightforwardly. For example, magnetization per unit area is given by

$$\mathcal{M} = \sum_i \left(-\mu_i \rho_i + \frac{eT}{mhc} \ln \frac{\rho_0 + \rho_i - \sum_j \alpha_{ij} \rho_j}{\rho_0 - \sum_j \alpha_{ij} \rho_j} \right). \quad (4.14)$$

Here $\mu_{\pm} = \partial \varepsilon_{\pm} / \partial B$; and we have assumed the independence of μ_{\pm} . Hopefully, when T is of the order of ε_{\pm} or higher, the α_{ij} -dependent second term may give an appreciable contribution. Note that the thermodynamic properties at the two sides of electron filling $\nu_0 \equiv N_e / N_{\phi} = 1/m$ are not symmetric, due to asymmetry of quasielectron and quasihole in self-exclusion and mutual-exclusion statistics.

V. EXPLICIT SOLUTIONS NEAR $\nu=1/m$

We present some explicit formulas for thermal activation of FQH quasiparticle pairs for three statistics matrices (3.13), (3.19), and (3.20), which were derived from three different counting schemes in Sec. III.

A. Bosonic vortex picture

From the statistics matrix (3.13), Eq. (4.6) determines the occupation number of quasielectrons and of quasiholes to be

$$n_+ = \frac{w_-}{(w_+ + 2 - 1/m)(w_- + 1/m) + 1/m^2}, \quad (5.1)$$

and

$$n_- = \frac{w_+ + 2}{(w_+ + 2 - 1/m)(w_- + 1/m) + 1/m^2}. \quad (5.2)$$

where w_+ and w_- can be obtained by solving Eqs. (4.9) and (4.10), which are explicitly

$$m \delta w_+ w_- + (\delta + 1) w_+ + [(2m - 1) \delta - 1] w_- + 2 + 2\delta = 0; \quad (5.3)$$

$$w_+^2 (1 + w_+)^{-1} (1 + w_-) = e^{\Delta/kT} \equiv f(T), \quad (5.4)$$

where $\Delta = \varepsilon_+ + \varepsilon_-$ is the pair excitation gap.

The last two equations cannot be solved analytically, but numerical solution is possible. See Figs. 1(a) and 2(a) for a two-dimensional plot of $n_+(T, \delta)$ and $n_-(T, \delta)$, obtained numerically for near $1/m = 1/3$.

B. Composite fermion scheme

In this scheme, the statistics matrix is given by Eq. (3.19). Then Eq. (4.6) is explicitly

$$n_+ = \frac{w_- + 1}{(w_+ + 2 - 1/m)(w_- + 1/m) + (1 - 1/m)^2}, \quad (5.5)$$

and

$$n_- = \frac{w_+ + 1}{(w_+ + 2 - 1/m)(w_- + 1/m) + (1 - 1/m)^2}, \quad (5.6)$$

with w_+ and w_- determined by

$$w_+ w_- = e^{\Delta/kT} \equiv f(T), \quad (5.7)$$

$$m \delta w_+ w_- + (\delta + 1) w_+ + [(2m - 1) \delta - 1] w_- + m \delta = 0. \quad (5.8)$$

In this case, the solution in analytic form is available: We obtain explicitly

$$\begin{aligned} w_+(T) = & \frac{1}{2(\delta + 1)} \{ -m \delta [f(T) + 1] \pm \sqrt{m^2 \delta^2 [f(T) + 1]^2 - 4(\delta + 1)[(2m - 1) \delta - 1] f(T)} \}, \\ w_-(T) = & \frac{f(T)}{w_+(T)}. \end{aligned} \quad (5.9)$$

The upper sign is for $\delta > 0$ and the lower sign for $\delta < 0$. At $T = 0$, Eq. (5.9) indeed yields $n_+ = m \delta, n_- = 0$, for $\delta > 0$, and similarly $n_+ = 0, n_- = -m \delta$, for $\delta < 0$, as expected from Eq. (4.8).

Numerical results for a two-dimensional plot of $n_+(T, \delta)$ and $n_-(T, \delta)$ for filling factors near $1/m = 1/3$ are shown in Figs. 1(b) and 2(b).

C. Correlated vortex or projected composite fermion picture

For the statistics matrix (3.13), Eq. (4.6) is explicitly

$$n_+ = \frac{w_- + 2}{(w_+ + 2 - 1/m)(w_- + 1/m) + (2 - 1/m)^2} \quad (5.10)$$

and

$$n_- = \frac{w_+}{(w_+ + 2 - 1/m)(w_- + 1/m) + (2 - 1/m)^2}, \quad (5.11)$$

with w_+ and w_- satisfying

$$m\delta w_+ w_- + (\delta + 1)w_+ + [(2m - 1)\delta - 1]w_- + 2(2m - 1)\delta - 2 = 0, \quad (5.12)$$

$$w_-^2(1 + w_+)(1 + w_-)^{-1} = f(T). \quad (5.13)$$

Again, analytic solution is impossible, but a numerical two-dimensional plot for $n_+(T, \delta)$ and $n_-(T, \delta)$ is shown, respectively, in Figs. 1(c) and 2(c), for filling factors near $1/m = 1/3$.

D. Low-temperature asymptotics

As an application of the above explicit formulas, let us discuss the low-temperature asymptotics of the density of activated pairs. For simplicity, we consider the case with exactly $\nu = 1/m$, or $\delta = 0$. It is easy to check that in either of the schemes, we have $n_+(T) = n_-(T)$. At very low temperatures, $f(T)$ is very large, so we have $w_{\pm} \approx \exp\{\Delta/2kT\}$. This leads to

$$\rho_{\pm}(T) \approx \rho_0 \exp\{\Delta/2kT\}, \quad (5.14)$$

with the prefactor $\rho_0 \equiv G_{\pm}/V = (1/m)N_{\phi}/V$, proportional to the (fractional) quasiparticle charge.

This is in complete agreement with the standard Boltzmann behavior characteristic of thermal activation across a finite gap. Note that this behavior is independent of the statistics matrix. Thus to look for the effects of fractional exclusion statistics, the temperature should be higher than this asymptotic region.

VI. EMERGENCE OF HIERARCHICAL STATES

From the general equation of state (4.11), one can easily see that the pressure P will diverge, when one of the denominators in the right side becomes zero, i.e., when the excitation densities satisfy

$$\sum_{j=+,-} \alpha_{ij} n_j(T) = 1 \quad (i = + \text{ or } -). \quad (6.1)$$

This corresponds to the situation in which one of the $G_{\text{eff},\pm}^{(b)}$ vanishes [see Eq. (2.5)], so that there is no available quasi-hole or quasielectron states for an additional thermally activated pair to occupy. Using Eq. (4.4), the condition (6.2) is reduced to

$$w_+ n_+ = 0 \text{ or } w_- n_- = 0. \quad (6.2)$$

At zero temperature in the ground state with the filling factor ν near the magic $\nu_0 = 1/m$ (m odd), there is no thermally activated pair, so one of the n_{\pm} vanishes, depending on whether ν is greater or smaller than $1/m$. There are three cases that the condition (6.2) is satisfied:

(i) $n_+ = n_- = 0$: This is the case with $\nu = \nu_0$, which is just the original Laughlin's $1/m$ incompressible state.

(ii) $n_- = 0$ and $n_+ > 0$: According to the relation (4.8), this is the case with $\nu > \nu_0$. Then $w_+ = 0$, and Eq. (4.10) requires $w_- = \infty$. So from Eq. (4.6), one obtains $n_+ = 1/\alpha_{++}$. For $\alpha_{++} = 2 - 1/m$, it leads to $\nu = 2/(2m - 1)$. This gives rise to a new incompressible state in which quasiholes are absent and quasielectrons fill up all possible states. In the bosonic vortex scheme, this new quantum Hall state is called the first hierarchical state,^{15,18} in which the hard-core quasielectrons in the Laughlin $1/m$ state are condensed to form a new incompressible liquid. If quasielectrons did not have a hard core, they would not be able to form the new incompressible hierarchical state. In the composite fermion scheme, the original Laughlin $1/m$ state corresponds to $n = 1$ in Eq. (3.14), and is interpreted as complete filling of the lowest Landau level in the residue magnetic field, while the new state corresponds to $n = 2$ with the second Landau level completely filled by composite fermions.

(iii) $n_+ = 0$ and $n_- > 0$: According to the relation (4.8), this is the case with $\nu < \nu_0$. Then $w_- = 0$, and Eq. (4.10) requires $w_+ = \infty$. So from Eq. (4.6), one obtains $n_- = 1/\alpha_{--}$. For $\alpha_{--} = 1/m$, it leads to $\nu = 1 - 1/m$ state, in conflict with the condition $\nu < \nu_0$. So unlike quasielectrons, the quasiholes in the Laughlin $1/m$ -liquid cannot condense to form a new incompressible state. Numerical data presented in Ref. 20 confirm this conclusion.

We note that in the above discussions for $T = 0$, mutual statistics is irrelevant.

What will happen at finite T ? Suppose that $\nu > \nu_0$. Due to thermal activation of quasiparticle pairs, both $n_{\pm}(T) > 0$, so there is an additional contribution in Eq. (6.1) from thermally activated quasiholes, which is T dependent. One may wonder if the effect of mutual statistics would lead to the formation of new incompressible (hierarchical) states at T -dependent filling factors. At finite T , the condition (6.2) is reduced to $w_+(T) = 0$ or $w_-(T) = 0$. It can be shown, case by case for the three statistics matrices (3.13), (3.19), and (3.20), that in either scheme, *the pressure cannot diverge at any filling factor*. For example, in the composite fermion scheme, Eq. (5.7) with finite T implies that if $w_+(T) = 0$, then $w_-(T) = \infty$, which in turn leads to $n_-(T) = 0$, in accordance to Eq. (5.6). But this is impossible at finite T due to thermal activation. Similarly $w_-(T) = 0$ would lead to $n_+(T) = 0$, a contradiction too. The same is true in the other two schemes.

The fact that the pressure P never diverges at finite T means that the quantum Hall transitions due to a divergent P are a quantum phase transition at zero temperature.

VII. THERMODYNAMIC OBSERVABLES

Under the assumption that quasiparticles dominate the low-lying excitation spectrum of the FQH liquids, the knowledge of thermal activation of quasiparticle pairs allows us to calculate thermodynamic observables of the FQH liquids. In this section, we will show numerical results with three possible statistics matrices, (3.13), (3.19), and (3.20).

In Fig. 3, we show the temperature dependence of the thermodynamic potential pV for several different filling factors.

The average energy density $h(T)$ of the quasielectron-quasihole pairs is given as

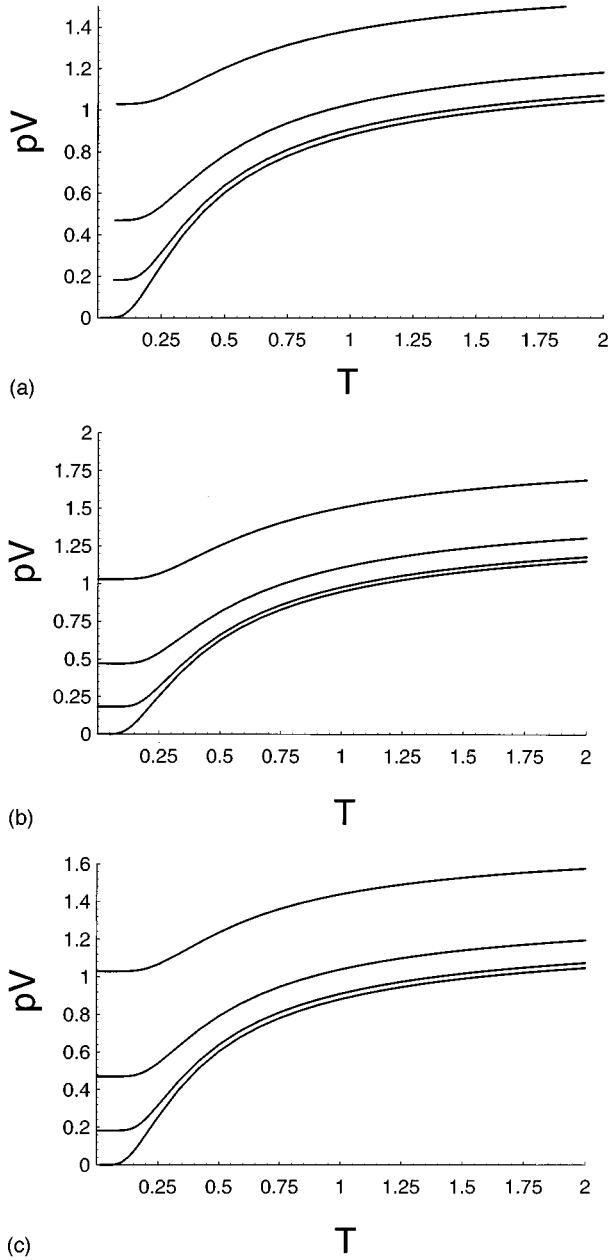


FIG. 3. The thermodynamic potential $\Omega = pV$ near the filling factor $1/3$. (a) For the bosonic vortex scheme, (b) For the composite fermion scheme, (c) For the correlated vortex or projected composite fermion scheme. In all the figures, $\delta = 0.15, 0.1, 0.05$, and 0 , respectively, from above.

$$h(T) = \varepsilon_+[(\rho_+(T) - \rho_+(0))] + \varepsilon_-[(\rho_-(T) - \rho_-(0))]. \quad (7.1)$$

The specific heat $C_v(T, \delta)$ is also evaluated numerically using the expression

$$C_v = \frac{\partial h(T)}{\partial T} = -k\rho_0 E^2 \frac{\partial n_-}{\partial E}, \quad (7.2)$$

where $E = \Delta/kT$. In Fig. 4, we have shown the curves of $C_v(T)$ at different values of δ for the three statistics matrix. At very low temperatures, C_v vanishes exponentially: $C_v(T) \sim \exp\{-\Delta/kT\}$, due to a finite activation gap. At higher temperatures, C_v increases due to quasiparticle pair creation.

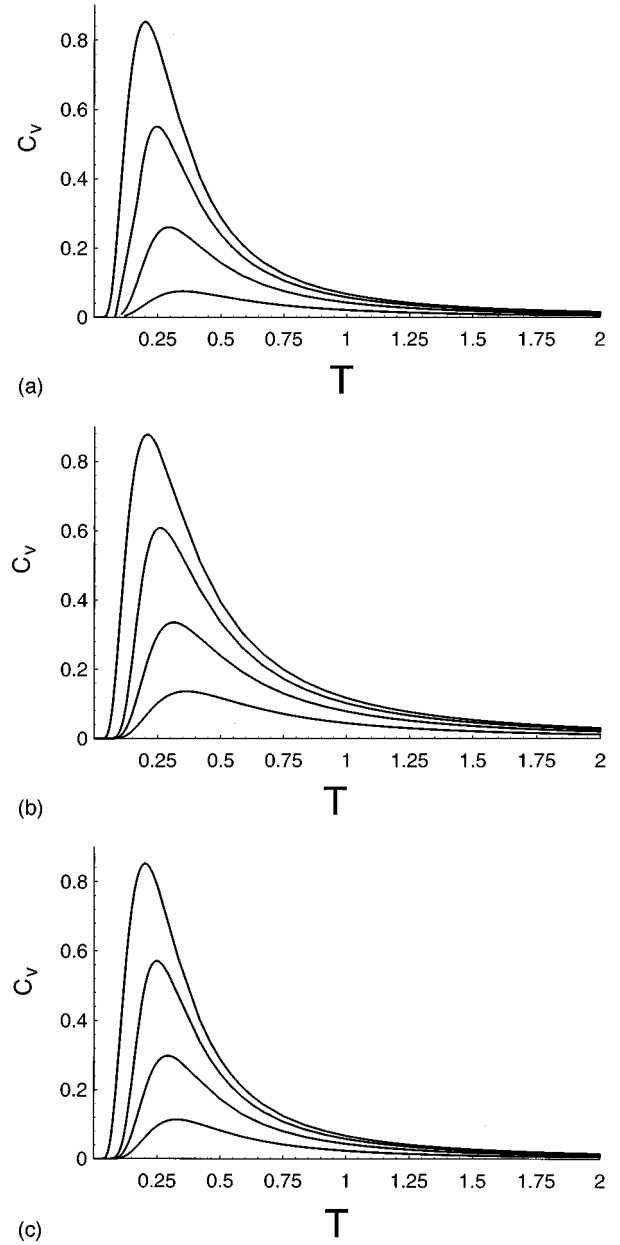


FIG. 4. The temperature dependence of the specific heat $C_v(T, \delta)$ near the filling factor $1/3$. The different curves are different δ 's. (From above, $\delta = 0, 0.05, 0.1, 0.15$.) (a) For the bosonic vortex scheme, (b) For the composite fermion scheme, (c) For the correlated vortex or projected composite fermion scheme. In all the figures, $\delta = 0, 0.05, 0.1$, and 0.15 , respectively, from above.

When the temperature approaches the order of magnitude of the gap energy Δ , the difference between different mutual statistics becomes more and more apparent. The phenomenon that $C_v(T)$ decreases after reaching a maximum is a manifestation of exclusion statistics due to the saturation of the available states for quasiparticles. The position of the maximum of the $C_v(T)$ curve is statistics-matrix dependent. Whether our approximation is still valid at this temperature or not is a question we cannot answer in our approach. We leave it to experiments. However, we are sure that the vanishing of C_v at very high temperatures should not be trusted, since our approximation of restricting quasielectrons to the lowest available Landau level certainly breaks down.

Now let us consider the magnetic response of the system. The magnetization per unit area due to quasiparticle excitations has been given in Ref. 5 as

$$\mathcal{M} = - \left(\frac{\partial \Omega / V}{\partial B} \right)_{T, V, \mu_i / kT} = \sum_{i=+, -} \left(-\mu_i^B \rho_i + \frac{kT}{3\phi_0} \ln \frac{1+w_i}{w_i} \right). \quad (7.3)$$

Here $\phi_0 = hc/e$. In the derivation, we have effectively defined $\mu_{\pm}^B = \partial \varepsilon_{\pm} / \partial B$. In comparison with experimental data, the effective magnetic moment, μ_{\pm}^B , of a single quasiparticle could be either treated as phenomenological parameters or derived from some microscopic model. In the following, we will assume μ_{\pm} to be B independent.

This equation can be viewed as a generalization of the case with one species of anyons in the lowest Landau level.^{31,5} The first term, corresponding to the usual de Haas–van Alphen term, is of the same form in the two cases. However, there are important differences between the one-species anyon case and the present situation: For the former case, the density of anyons, ρ , is fixed either when B or T varies (no thermal activation of anyon pairs). The resulting susceptibility has a simple analytic expression and vanishes as $T \rightarrow 0$.⁵ However, this is not true in the present case, where the densities of quasielectrons and quasiholes are both B and T dependent. Even at zero temperature, the magnetization has a B dependence through that of ρ_{\pm} :

$$\mathcal{M}(T=0) = \mu_{\pm}^B \rho_{\pm}(B, T=0) = \mu_{\pm}^B \rho_0 n_{\pm}(B, T=0). \quad (7.4)$$

(The subscript \pm depends on whether $\nu - 1/m$ is positive or negative.) This implies that the zero-temperature susceptibility is nonzero, in contrast to the case of anyons of one species:

$$\chi(T=0) = \left(\frac{\partial \mathcal{M}(T=0)}{\partial B} \right)_V = \frac{\mu_{+}^B}{\phi_0}. \quad (7.5)$$

Our general expression for the finite-temperature susceptibility reads

$$\chi = \left(\frac{\partial \mathcal{M}}{\partial B} \right)_{T, V} = \chi_0 + \chi_1, \quad (7.6)$$

where χ_0 is a de Haas–van Alphen–like term and χ_1 comes from thermally activated pairs. They are given by

$$\begin{aligned} \chi_0 &= \frac{\mu_{+}^B}{3\phi_0} \sum_{i=+, -} \left(-n_i + (\delta+1) \frac{\partial n_i}{\partial \delta} \right), \\ \chi_1 &= \frac{\Delta}{9\rho_e \phi_0^2} \sum_{i=+, -} \frac{(\delta+1)^2}{E w_i (1+w_i)} \frac{\partial w_i}{\partial \delta}, \end{aligned} \quad (7.7)$$

where we have assumed $\mu_{+}^B = \mu_{-}^B$ for simplicity. Figures 5 and 6 show the susceptibilities χ_0 and χ_1 as functions of T and $\delta = 3\nu - 1$ (with $0 \leq \delta \leq 1/5$ corresponding to $1/3 \leq \nu \leq 2/5$), for three different counting schemes for statistics matrix. Indeed, for $1/3 < \nu < 2/5$, $\chi_1 \propto T$ and tends to zero as $T \rightarrow 0$, so that $\chi \rightarrow \chi_0(T=0)$ as expected. When $\Delta \sim 100$ mK, $\rho_e \sim 10^{11} \text{ cm}^{-2}$ and $\mu_{+}^B \sim \mu_0^B$ (Bohr's magnetic

moment), the ratio of the coefficients of χ_0 , $A_0 = \mu_{+}^B / 3\phi_0$, and that of χ_1 , $A_1 = \Delta / 9\rho_e \phi_0^2$, is $A_0 / A_1 \sim 10^2$. Therefore, in a certain range of filling factor greater than $\nu_0 = 1/3$, the de Haas–van Alphen term dominates, which is positive at $T=0$. On the other hand, from Fig. 5, we see that χ_1 is always negative in the range of fillings at hand. As T increases, the magnitude of χ_1 increases, gradually becomes comparable to χ_0 and eventually dominates. Then the total susceptibility will change sign. Thus, we observe that for a given filling factor, there can be a paramagnetism-diamagnetism transition, with the critical temperature generically $T_c \sim \Delta$. The new incompressible state with $\nu = 2/5$ at $T=0$ is totally diamagnetic, because with $\nu \rightarrow 2/5$ one has $w_{-} \rightarrow 0$, which causes χ_1 to be divergent. This might be viewed as a generalization of Landau diamagnetism to quasielectrons.

VIII. CONCLUSIONS AND DISCUSSIONS

Quasielectrons and quasiholes in the FQH liquids obey fractional (including nontrivial mutual) exclusion statistics. Their statistics matrix near the magic electron filling $1/m$ (with m odd) can be determined from various state-counting schemes. These schemes involve different assumptions on statistical correlations between quasiparticles, resulting in somewhat different mutual statistics. A common feature of the schemes is that both quasiholes and quasielectrons in the incompressible FQH liquid background behave like a charge in an effective magnetic field. The common assumption is that quasielectrons are in the lowest available Landau level (with respect to the effective magnetic field). Then the thermal activation of FQH quasiparticles at low temperatures is discussed in the dilute generalized ideal gas approximation. If these quasiparticles dominate the low-lying excitation spectrum, their contributions dominate the low-temperature thermodynamics. Otherwise, contributions from other low-lying elementary excitations (such as skyrmions, if they exist and are important) have to be added to the quasiparticle contributions we obtain here. At higher temperatures, our assumptions obviously break down. We hope the situation could be improved in the future by incorporating corrections. Right now we just leave the question of when the corrections should be important open to experiments.

We have used the approximation of a generalized ideal gas in our treatment of statistical thermodynamics. It is good only when the thermally activated quasiparticle pairs are not too dense. On the other hand, to look for the effects of mutual statistics, the densities of quasiparticle pairs should not be too low. We hope there is some intermediate range for quasiparticle densities in which the two conflicting requirements could be reconciled to some extent. Whether this is true only experiments can tell.

It is shown that the existence of hierarchical states in the FQH effect can be viewed as a manifestation of the exclusonic nature of the relevant quasiparticles. The associated FQH phase transition is shown to occur only at zero temperature. Thermal activation of quasiparticle pairs and thermodynamic observables is numerically studied with three possible statistical matrices. At zero temperature, they are all equivalent to each other, but differences show up at finite temperature. In particular, we have demonstrated that for a fixed filling factor between $1/3$ and $2/5$, with increasing temperature, the

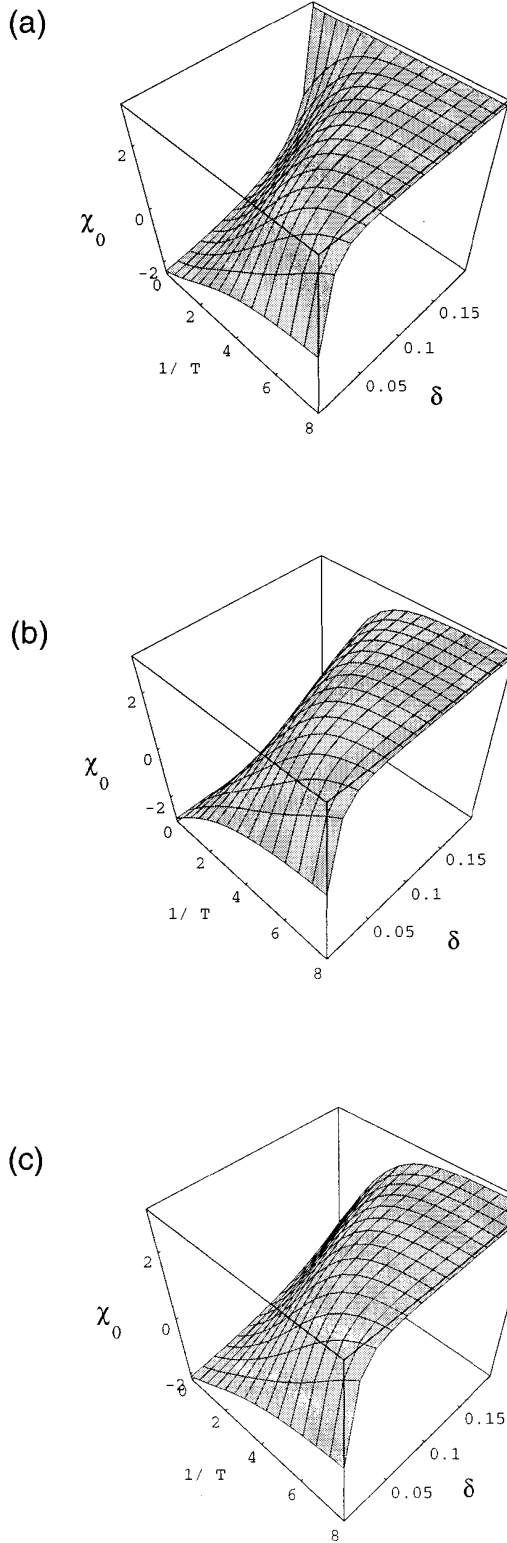


FIG. 5. The two-dimensional plot of $\chi_0(T, \delta)$ (de Haas-van Alphen term of the susceptibility) near the filling factor $1/3$. (a) For the bosonic vortex scheme, (b) for the composite fermion scheme, (c) for the correlated vortex or projected composite fermion scheme.

system may possibly exhibit a transition from paramagnetism to diamagnetism. However, we should be cautious about this possibility: We have assumed that μ_{\pm}^B are of the same order of magnitude as the Bohr magneton; also the approxi-

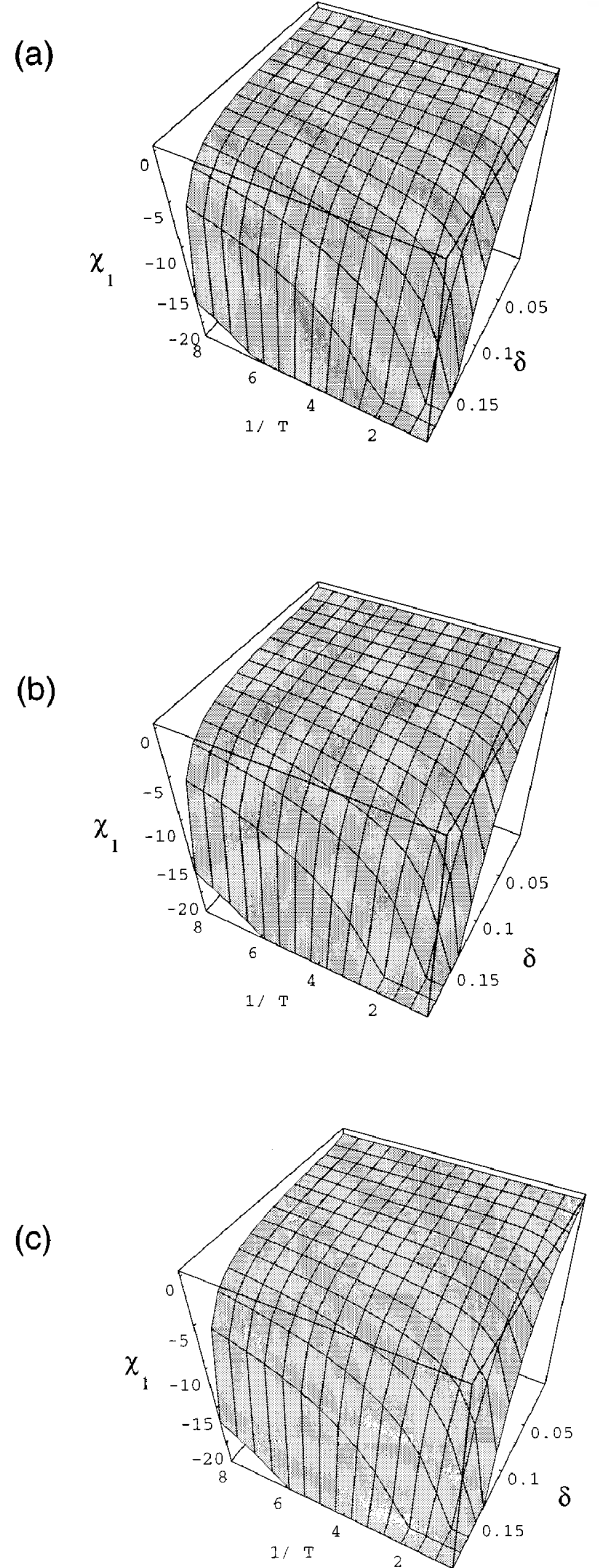


FIG. 6. The two-dimensional plot of $\chi_1(T, \delta)$ (a pair excitation term of the susceptibility) near the filling factor $1/3$. (a) For the bosonic vortex scheme, (b) for the composite fermion scheme, (c) for the correlated vortex or projected composite fermion scheme.

mation of the generalized ideal gas might break down at the would-be transition temperature.

To conclude, we have found that what are usually thought of as different but equivalent pictures of the fractional quan-

tum Hall effect (anyon, composite fermion, and composite boson) do not lead to the same exclusion statistics matrix, with the difference essentially in the off-diagonal or mutual statistics. Assuming that the quasiparticles can be modeled as weakly interacting exclusonic quasiparticles, we have explored the differences in thermodynamics resulting from the different possible schemes. It is not entirely surprising that the different schemes lead to different exclusonic statistics, since the quasiparticles are known to have different exchange (or braiding) statistics in the different schemes. (The different schemes in exchange statistics agree when it comes to any physical measurement, which involves the combination of exchange statistics and interactions, which also differ between the schemes.) Presumably, it is possible for the quasiparticles to be modeled as weakly interacting exclusonic particles in only one of these schemes, if at all. By comparing the results of this paper with numerics or experiments, it

should be possible to determine which (if any) scheme allows for such a weakly interacting description. In particular, it is more desirable that these theoretical predictions would be put to experimental tests, if the tremendous difficulties in measuring thermodynamic quantities of a thin layer of electron gas could be overcome someday.

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